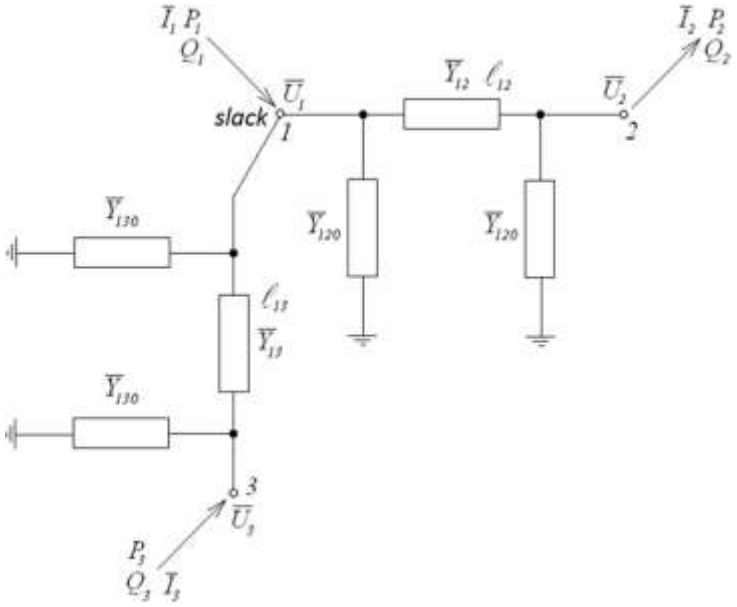
Example: Analytically solve the voltage conditions in the 3-bus network (see below), where bus no. 1 is the slack bus ($\overline{U}_1 = 110000 \cdot e^{j0}$ V). Remaining two buses are of PQ type: $P_2 = 50$ MW, $Q_2 = 10$ MVAr, $P_3 = 100$ MW, $Q_3 = 10$ MVAr.
Network buses are interconnected by power lines with the following parameters: $X = 0.4\ \Omega/\text{km}, B = 3\ \mu\text{S/km}, \ell_{12} = 100$ km, $\ell_{13} = 50$ km.
Resistances and shunt conductances are neglected.



Calculation:

The task is to analytically (i.e. without the use of numerical methods) find the values of $U_2, \vartheta_2, U_3, \vartheta_3$.

1) Calculation of passive parameters:
Line reactance: $X_{12} = X \cdot \ell_{12} = 0.4 \cdot 100 = 40\ \Omega, X_{13} = X \cdot \ell_{13} = 0.4 \cdot 50 = 20\ \Omega$
Line susceptance: $B_{120} = B \cdot \frac{\ell_{12}}{2} = 3\mu\frac{100}{2} = 150\ \mu\text{S}, B_{130} = B \cdot \frac{\ell_{13}}{2} = 3\mu\frac{50}{2} = 75\ \mu S$
Series admittance:

$$\overline{Y}_{12} = \frac{R_{12}}{{R_{12}}^2 + {X_{12}}^2} - j\frac{X_{12}}{{R_{12}}^2 + {X_{12}}^2} = -j0.025\ \text{S}$$

$$\overline{Y}_{13} = \frac{R_{13}}{{R_{13}}^2 + {X_{13}}^2} - j\frac{X_{13}}{{R_{13}}^2 + {X_{13}}^2} = -j0.05\ \text{S}$$

Shunt admittance:

$$\overline{Y}_{120} = G_{120} + jB_{120} = j150\ \mu\text{S}, \overline{Y}_{130} = G_{130} + jB_{130} = j75\ \mu S$$

2) Admittance matrix:

$$\underline{\overline{A}} = \begin{bmatrix} \overline{Y}_{12} + \overline{Y}_{120} + \overline{Y}_{13} + \overline{Y}_{130} & -\overline{Y}_{12} & -\overline{Y}_{13} \\ -\overline{Y}_{12} & \overline{Y}_{12} + \overline{Y}_{120} & 0 \\ -\overline{Y}_{13} & 0 & \overline{Y}_{13} + \overline{Y}_{130} \end{bmatrix} = j \begin{bmatrix} -0.074775 & 0.025 & 0.05 \\ 0.025 & -0.02485 & 0 \\ 0.05 & 0 & -0.049925 \end{bmatrix}$$

3) Final set of load flow equations in matrix form:

Injected currents in each network bus are:

$$\overline{I}_1 = \frac{P_1 - jQ_1}{\sqrt{3}\overline{U}_1^*}, \overline{I}_2 = \frac{-(P_2 - jQ_2)}{\sqrt{3}\overline{U}_2^*} = \frac{-(50 - j10)\cdot 10^6}{\sqrt{3}\overline{U}_2^*}, \overline{I}_3 = \frac{P_3 - jQ_3}{\sqrt{3}\overline{U}_3^*} = \frac{(100 - j10)\cdot 10^6}{\sqrt{3}\overline{U}_3^*}$$

According to sign agreement, power P_2 and Q_2 flowing out from bus no. 2 are negative. Final set of load flow equations in matrix form is shown below.

$$\begin{bmatrix} \frac{P_1 - jQ_1}{\overline{U}_1^*} \\ \frac{-(50 - j10)\cdot 10^6}{\overline{U}_2^*} \\ \frac{(100 - j10)\cdot 10^6}{\overline{U}_3^*} \end{bmatrix} = j \begin{bmatrix} -0.074775 & 0.025 & 0.05 \\ 0.025 & -0.02485 & 0 \\ 0.05 & 0 & -0.049925 \end{bmatrix} \cdot \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \end{bmatrix}$$

4) Load flow calculation

The set of 3 equations contains six unknown variables $(U_2, \vartheta_2, U_3, \vartheta_3, P_1, Q_1)$. For the calculation of the remaining two line-to-line bus voltages we exclude the equation for the slack bus voltage (row no. 1). Remaining rows look as follows:

$$\frac{-(50 - j10)\cdot 10^6}{\overline{U}_2^*} = j0.025 \cdot \overline{U}_1 - j0.02485 \cdot \overline{U}_2$$

$$\frac{(100 - j10)\cdot 10^6}{\overline{U}_3^*} = j0.05 \cdot \overline{U}_1 - j0.049925 \cdot \overline{U}_3$$

4a) Solving the first equation for bus voltage $\overline{U}_2$:

$$\frac{-(50 - j10)\cdot 10^6}{\overline{U}_2^*} = j0.025 \cdot 110000 - j0.02485 \cdot \overline{U}_2$$

Multiply the whole equation by $\overline{U}_2^*$ and convert $\overline{U}_2^*$ into complex form:

$$-50 \cdot 10^6 + j10 \cdot 10^6 = j0.025 \cdot 110000 \cdot U_2 \cdot (\cos\vartheta_2 - j\sin\vartheta_2) - j0.02485 \cdot {U_2}^2$$

Next step is the comparison of real and imaginary parts of complex values on both sides of the equation:

$$-50 \cdot 10^6 = 2750 \cdot U_2 \cdot \sin\vartheta_2$$

$$10 \cdot 10^6 = 2750 \cdot U_2 \cdot \cos\vartheta_2 - 0.02485 \cdot {U_2}^2$$

Now we have the set of two equations for two unknowns (U_2, ϑ_2), which must be solved. Although the set of equations seems nonlinear, the analytical approach is possible for use. First, the second equation is squared and term „$cos^2 \vartheta_2$" is replaced by term „$1 - sin^2 \vartheta_2$".

After changes we get:

$$1 \cdot 10^{14} + 497000 \cdot {U_2}^2 + 6.175225 \cdot 10^{-4} \cdot {U_2}^4 = 7562500 \cdot {U_2}^2 - 7562500 \cdot {U_2}^2 \cdot sin^2 \vartheta_2$$

From first equation we separate the term „$sin \vartheta_2$" and put it into the newly obtained equation above. After few changes we get the biquadratic equation for the magnitude of line-to-line voltage $\overline{U}_2$:

$$6.175225 \cdot 10^{-4} \cdot {U_2}^4 - 7065500 \cdot {U_2}^2 + 2.6 \cdot 10^{15} = 0$$

By solving this equation we get two solution candidates, where the first one is the correct solution. The second is nonsense, because the final voltage must be close to nominal value of $\overline{U}_1$ (the power loads are reasonable and realistic, therefore there is no reason for such high voltage drop).

1. candidate: $U_2 = 105171.4763$ V
2. candidate: $U_2 = 19510.2265$ V (nonsense)

By putting the final value into the first equation, we get the phase angle ϑ_2:

$$sin \vartheta_2 = -0.17288 \quad \rightarrow \quad \vartheta_2 = -9°57'18.67''$$

We found the value of line-to-line voltage in bus no. 2:

$$\overline{U}_2 = 105171.4763 e^{-j9°57'18.67''} \text{ V}.$$

4b) Solving the second equation for bus voltage $\overline{U}_3$ (analogically):

$$\frac{(100 - j10) \cdot 10^6}{\overline{U}_3^*} = j0.05 \cdot 110000 - j0.049925 \cdot \overline{U}_3$$

$$100 \cdot 10^6 - j10 \cdot 10^6 = j0.05 \cdot 110000 \cdot U_3 \cdot (cos \vartheta_3 - j \, sin \vartheta_3) - j0.049925 \cdot {U_3}^2$$

The set of equations for bus voltage $\overline{U}_3$:

$$100 \cdot 10^6 = 5500 \cdot U_3 \cdot sin \vartheta_3$$
$$-10 \cdot 10^6 = 5500 \cdot U_3 \cdot cos \vartheta_3 - 0.049925 \cdot {U_3}^2$$

Then:

$$1 \cdot 10^{14} - 998500 \cdot {U_3}^2 + 2.492505625 \cdot 10^{-3} \cdot {U_3}^4 = 30250000 \cdot {U_3}^2 - 30250000 \cdot {U_3}^2 \cdot sin^2 \vartheta_3$$

Final biquadratic equation:

$$2.492505625 \cdot 10^{-3} \cdot {U_3}^4 - 31248500 \cdot {U_3}^2 + 1.01 \cdot 10^{16} = 0$$

Solution:

1. candidate: $U_3 = 110476.1282$ V
2. candidate: $U_3 = 18221.08223$ V (nonsense)

Phase angle of voltage $\overline{U}_3$ will be:

$$sin\,\vartheta_3 = 0.164577 \qquad \rightarrow \qquad \vartheta_3 = 9°28'21.56''$$

The line-to-line voltage in bus no. 3 is: $\overline{U}_3 = 110476.1282e^{j9°28'21.56''}$ V.

In this simple 3-bus network example it can be seen how much complicated is the analytical approach. When including series resistance and shunt conductance, or when choosing another network bus as the slack bus, or when solving larger networks, the analytical approach is not possible at all.